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On some integrals of Glauert's type

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Abstract

Some principal value integrals of Glauert's type are calculated analytically. These integrals appear when dealing with the hydrodynamic impact problem of a two-dimensional wedge onto a liquid free surface in the potential theory.

Keywords: Analytic integration; Hydrodynamics; Glauert's integrals

There are many situations in aerodynamics and in hydrodynamics as well, where the so-called Glauert's integral must be calculated. Its usual expression is

$$PV \int_0^{\pi} \frac{\cos k\theta}{\cos\theta - \cos\alpha} \mathrm{d}\theta = \pi \frac{\sin |k|\alpha}{\sin\alpha}, \tag{1}$$

where PV means the Principal Value of the integral. As recalled in [1], the usual way to calculate these integrals is to perform the integration of the following function in the complex plane z

$$f(z) = \frac{z^n}{z^2 - 2z\cos\alpha + 1}, \tag{2}$$

where z describes the unit circle $z = \mathrm{e}^{\mathrm{i}\theta}$. When studying the impact of an elastic two-dimensional beam or wedge onto a liquid free surface within the Wagner [2] approach (potential theory), we face the usual "lifting" problem which must be turned into a "thickness" problem (see [3, pp. 180–184]). Applications

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of such problems are described in [4]. As a result in the frame of a modal-based method, we end up with the computation of the following integral

$$I_k(\alpha) = PV \int_{-\pi/2}^{\pi/2} \frac{sg(\theta)\sin(2k+1)\theta}{\sin\theta - \sin\alpha} \mathrm{d}\theta = -4\sin\alpha PV \int_0^{\pi/2} \frac{\sin(2k+1)\theta}{\cos 2\theta - \cos 2\alpha} \mathrm{d}\theta, \tag{3}$$

where $\alpha \in [0, \pi/2]$ and $k \geq 0$. Integral (3) looks like a Glauert's integral but its calculation is more delicate due to the fact that the integrand has no parity and also the interval of integration is not the whole unit circle. It is difficult to calculate directly the integral $I_k(\alpha)$ by using a contour method in the complex plane, unless the same artifice – as presented here – is utilized. An alternate calculation of integral (3) where only a finite summation is necessary is proposed here. As a starting point the following identity is introduced

$$\sin(2k+1)\theta = \sin(2k-1)\theta + 2\sin\theta\cos 2k\theta, \tag{4}$$

to express $I_k(\alpha)$ as a recursive scheme

$$I_k(\alpha) = I_{k-1}(\alpha) - 8M_k(\alpha)\sin\alpha, \quad \text{with } M_k(\alpha) = PV \int_0^{\pi/2} \frac{\sin\theta\cos 2k\theta}{\cos 2\theta - \cos 2\alpha} \,\mathrm{d}\theta, \tag{5}$$

and the initial value

$$I_0(\alpha) = -4M_0(\alpha)\sin\alpha. \tag{6}$$

In order to calculate $M_k(\alpha)$, the key identity is used

$$sg(\alpha)\sin\alpha = \sum_{\ell=0}^{\infty} B_\ell \cos 2\ell\alpha, \quad B_\ell = \frac{4}{\pi}\begin{cases} \dfrac{1}{2} & \ell = 0 \\ \dfrac{1}{1-4\ell^2} & \ell > 0. \end{cases} \tag{7}$$

By using Glauert integrals, $M_k(\alpha)$ is written as a series

$$M_k(\alpha) = M_0(\alpha)\cos 2k\alpha + \frac{\pi}{2\sin 2\alpha}\sum_{\ell=0}^{k} B_\ell \sin 2(k-\ell)\alpha, \tag{8}$$

with

$$M_0(\alpha) = -\frac{S_c(\alpha)}{\cos\alpha}, \quad \text{and} \quad S_c(\alpha) = \frac{1}{2}\log\left(\frac{\sin\alpha}{1-\cos\alpha}\right). \tag{9}$$

Knowing that the double summation reduces according to

$$\sum_{i=1}^{k}\sum_{m=1}^{i} B_{i-m}\frac{\sin 2m\alpha}{\cos\alpha} = 2\sum_{m=0}^{k-1} B_m \frac{\sin(k-m+1)\alpha\sin(k-m)\alpha}{\sin 2\alpha}, \tag{10}$$

finally the original integral has the form

$$I_k(\alpha) = 2\frac{\sin(2k+1)\alpha}{\cos\alpha}\log\left(\frac{\sin\alpha}{1-\cos\alpha}\right) - 2\pi\sum_{\ell=1}^{k} B_{k-\ell}\left[\frac{\sin^2\ell\alpha}{\sin\alpha} + \frac{\sin 2\ell\alpha}{2\cos\alpha}\right], \quad k \geq 1. \tag{11}$$

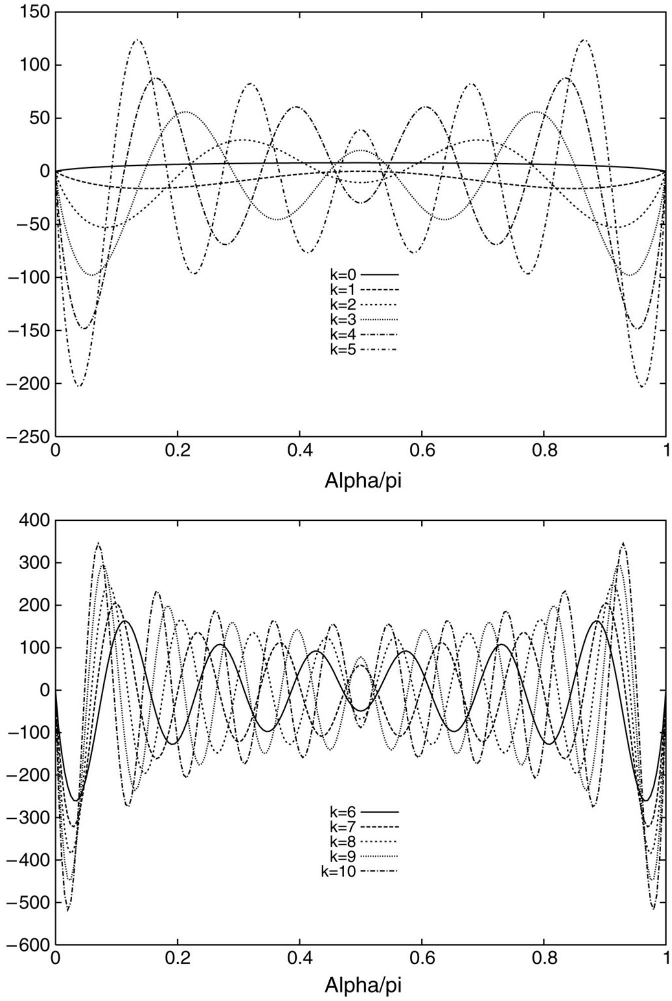

Fig. 1. Variation of $I_k(\alpha)$ in the interval $\alpha \in [0, \pi/2]$ and $k \in \{0, 1, 2, 3, 4, 5\}$ (up), $k \in \{6, 7, 8, 9, 10\}$ (below).

The integral $I_0(\alpha)$ is given by Eq. (6). The limiting values of $I_k(\alpha)$ at the origin $\alpha = 0$ and at $\alpha \to \frac{\pi}{2}$ are easily extracted. Fig. 1 shows the variation of $I_k(\alpha)$ for the first eleven values of k.

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